JOURNAL OF APPROXIMATION THEORY **18**, 13–29 (1976)

Jackson–Müntz–Szász Theorems in $L^p[0, 1]$ and $C[0, 1]$ for Complex Exponents

MANFRED V. GOLITSCHEK*

Institut für Angewandte Mathematatik, Universität Würzburg, Würzburg, BRD

Communicated by G. G. Lorentz

Received December 28, 1974

INTRODUCTION

Let $C[0, 1]$ be the space of all complex valued continuous functions with the norm

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|,$$

and $L^p[0, 1]$, $1 \leqslant p < \infty$, be the space of all complex valued measurable functions f, for which

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$$

is finite. The famous theorem of K. Weierstrass [18] states that the monomials $\{1, x, x^2, ...\}$ are a fundamental sequence in $C[0, 1]$, that is, a sequence of elements whose linear combinations are dense in $C[0, 1]$. This theorem has been generalized in two different directions by C. Müntz [13], O. Szász [16], and D. Jackson [8].

Müntz's theorem states that a sequence of monomials $\{1, x^{\lambda_1}, x^{\lambda_2}, ...\}$ of a real positive increasing sequence $\{\lambda_k\}_{k=1}^\infty$ is fundamental in $C[0, 1]$ if and only if $\sum_{k=1}^\infty 1/\lambda_k$ diverges. Müntz's theorem and its L^p analog have been extended for complex exponents λ_k in the following theorem and its corollary.

THEOREM (O. Szász). *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be a sequence of distinct complex numbers with real parts exceeding $-\frac{1}{2}$. Then the functions $\{x^{\lambda_1}, x^{\lambda_2}, ...\}$ are fundamental in $L^2[0, 1]$ if and only if*

$$\sum_{k=1}^\infty [(1 + 2\text{Re}\, \lambda_k)/(1 + |\lambda_k|^2)] = \infty.$$

* The Deutsche Forschungsgemeinschaft has sponsored this research under Grant No. GO 270/1.

Let the real parts of all numbers λ_k $(k = 1, 2, \ldots)$ *be positive. Then the functions* $\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *are fundamental in* $C[0, 1]$, *if*

$$\sum_{k=1}^{\infty} [\operatorname{Re} \lambda_k/(1 + |\lambda_k|^2)] = \infty,$$

and are not fundamental in $C[0, 1]$, *if*

$$\sum_{k=1}^{\infty} [(1 + \operatorname{Re} \lambda_k)/(1 + |\lambda_k|^2)] < \infty.$$

(For the proof compare also R. Paley and N. Wiener [15, Chap. II].)

As the continuous functions are dense in $L^p[0, 1]$, $1 \leqslant p < \infty$, we easily obtain the following. (We write $L^\infty[0, 1] = C[0, 1]$.)

COROLLARY. *Let* Λ *be a sequence of distinct complex numbers with real parts exceeding a positive number* ϵ. *Then the functions* $\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *are fundamental in* $L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, *if and only if*

$$\sum_{k=1}^{\infty} (\operatorname{Re} \lambda_k/|\lambda_k|^2) = \infty. \tag{1}$$

THEOREM (D. Jackson). *For each function* $f \in C[0, 1]$, *there exists an ordinary algebraic polynomial* P_n *of degree* n *such that*

$$\| f - P_n \|_\infty \leqslant K w_\infty(f, 1/n), \tag{2}$$

where K *is an absolute real constant and*

$$w_\infty(f; \delta) = \sup_{|t| \leqslant \delta} \| f(x + t) - f(x) \|_\infty, \qquad 0 \leqslant \delta \leqslant 1,$$

denotes the modulus of continuity of f.

The above Jackson theorem holds also for all L^p spaces, $1 \leqslant p < \infty$, if in (2) the modulus of continuity is replaced by the analogous L^p modulus of continuity

$$w_p(f; \delta) = \sup_{|t| \leqslant \delta} \| f(x + t) - f(x) \|_p, \qquad 0 \leqslant \delta \leqslant 1, f \in L^p[0, 1],$$

where we continue f by $f(x) = f(-x)$ for $-1 \leqslant x < 0$, $f(x) = f(2 - x)$ for $1 < x \leqslant 2$. (The theorems of Jackson and Müntz and some other results we have to apply are usually proved for real valued functions f and real coefficients. It is easy to verify that they are also valid in the complex case).

In recent years D. Newman [14], J. Bak and D. Newman [2, 3], T. Ganelius and S. Westlund [4], D. Leviatan [10], and the author [5, 6] combined the theorems of Jackson and Müntz and found several best or almost best "Jackson–Müntz theorems" for Λ-polynomials with real exponents Λ. In this paper we combine the theorems of Jackson and Szász and obtain the corresponding "Jackson–Müntz–Szász theorems" for Λ-polynomials with complex exponents Λ. All results of my earlier papers and almost all results of the other authors mentioned above can be derived easily as special cases.

1. The Basic Method

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ denote a sequence of distinct complex numbers with positive real parts. For $f \in L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, let

$$E_s(f; \Lambda)_p = \inf_{b_k \in \mathbb{C}} \left\| f(x) - b_0 - \sum_{k=1}^{s} b_k x^{\lambda_k} \right\|_p$$

be the degree of best approximation of f in $L^p[0, 1]$ by Λ-polynomials of "degree" s. For each ordinary algebraic polynomial

$$P_n(x) = \sum_{q=0}^{n} a_{qn} x^q$$

we obtain an upper bound for $E_s(f; \Lambda)_p$, if we replace each monomial x^q $(q = 1, 2,..., n)$ of P_n by its best Λ-polynomial of degree s. Thus

$$E_s(f; \Lambda)_p \leqslant \| f - P_n \|_p + \sum_{q=1}^{n} | a_{qn} | \, E_s(x^q; \Lambda)_p \, . \tag{3}$$

This is the essential idea. To apply the inequality (3) efficiently (given Λ, p, f, and s) we have to find an appropriate integer n depending on s and a good approximating polynomial P_n with relatively small coefficients a_{qn} $(q = 1,..., n)$. Such polynomials are provided in the following.

Lemma 1. *For any function $f \in L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, and any $n \geqslant 1$ there exists an even polynomial P_n such that*

$$\| f - P_n \|_p \leqslant C_p w_p(f; 1/n), \tag{4}$$

$$| a_{qn} | \leqslant D_p w_p(f; 1/n) \, n^{q+1/p}/q! \, , \qquad q = 1, 2,..., n, \tag{5}$$

$(a_{2k+1,n} = 0$ for $k = 0, 1,...)$, where C_p and D_p are absolute constants.

Proof. We define the even function $F \in L^p[-2, 2]$ by

$$F(x) = \begin{cases} f(x) & \text{for } 0 \leqslant x \leqslant 1, \\ f(2 - x) & \text{for } 1 \leqslant x \leqslant 2. \end{cases}$$

Then Jackson's theorem in $L^p[-2, 2]$, $1 \leqslant p \leqslant \infty$, states that there exists for any $m \geqslant 1$ an even polynomial P_m, for which

$$\| F - P_m \|_{L^p[-2,2]} \leqslant C_p' w_p(F; 1/m) \tag{6}$$

is satisfied, where C_p' is an absolute constant and $w_p(F; \cdot)$ refers to the interval $[-2, 2]$. We write $w(1/m) = C_p' w_p(F; 1/m)$ and define the integer t by $2^t < n \leqslant 2^{t+1}$. For any integers n_1, n_2 with $1 \leqslant n_1 < n_2 \leqslant 2n_1$, it follows from a result of G. K. Lebed' [9] that

$$\| P_{n_2} - P_{n_1} \|_{C[-1,1]} \leqslant D_p' n_2^{1/p} \| P_{n_2} - P_{n_1} \|_{L^p[-2,2]},$$

where D_p' is an absolute constant. Using (6) we therefore obtain

$$\| P_{n_2} - P_{n_1} \|_{C[-1,1]} \leqslant 2D_p' n_2^{1/p} w(1/n_1).$$

Finally, applying an inequality of A. F. Timan [17, 4.8.81] we have, for $q = 1, 2, \ldots, n$,

$$| a_{qn_2} - a_{qn_1} | \leqslant 2D_p' n_2^{q+1/p} w(1/n_1)/q!. \tag{7}$$

As $w_p(F; \delta) \leqslant C_p'' w_p(f; \delta)$, $0 \leqslant \delta \leqslant 1$, we conclude from (6) that the polynomial P_n satisfies (4). Moreover, the coefficients $a_{2k+1,n} = 0$ ($k = 0, 1, \ldots$) since P_n is even. Applying (7) and the inequality

$$| a_{qn} | \leqslant | a_{qn} - a_{q2^t} | + \sum_{j=1}^{t} | a_{q2^j} - a_{q2^{j-1}} | + | a_{q1} |$$

for all even indices $q = 2, 4, \ldots$ we obtain (5). Thus, the proof of Lemma 1 is complete.

In our next Lemma we give upper bounds for the best approximations

$$\tilde{E}_s(x^q; \Lambda)_p = \inf_{a_k \in \mathbb{C}} \left\| x^q - \sum_{k=1}^{s} a_k x^{\lambda_k} \right\|_p \text{ or } E_s(x^q; \Lambda)_p$$

of the monomials x^q, where q may be any real number exceeding $-1/p$. (Analogous results for complex numbers q are also valid.) For the L^p norms with $1 \leqslant p < 2$ we have inserted a positive number ϵ. This is perhaps unnecessary, but we can only prove the inequality (11).

LEMMA 2. *Let Λ be a sequence of complex numbers with real parts exceeding $-1/p$. Then, for any real number $q > -1/p$ and any integer $s \geqslant 1$,*

$$\tilde{E}_s(x^q; \Lambda)_2 = \frac{1}{(2q+1)^{1/2}} \prod_{k=1}^{s} \frac{| q - \lambda_k |}{| q + \bar{\lambda}_k + 1 |}; \tag{8}$$

$$\tilde{E}_s(x^q; \Lambda)_\infty \leqslant \prod_{k=1}^{s} \frac{|q-\lambda_k|}{|q+\bar{\lambda}_k|}; \tag{9}$$

$$E_s(x^q; \Lambda)_p \leqslant A_p \frac{|q|}{(2q+2/p)^{1/2}} \prod_{k=1}^{s} \frac{|q-\lambda_k|}{|q+\bar{\lambda}_k+2/p|} \tag{10}$$

for $2 < p < \infty$, *where* $A_p = (1+p/2)^{1/2+1/p}$;

$$\tilde{E}_s(x^q; \Lambda)_p \leqslant \frac{\epsilon^{-(2-p)/(2p)}}{(2q+2(1-\epsilon)/p)^{1/2}} \prod_{k=1}^{s} \frac{|q-\lambda_k|}{|q+\bar{\lambda}_k+2(1-\epsilon)/p|} \tag{11}$$

for $1 \leqslant p < 2$ *and any* $0 < \epsilon < 1 + pq$.

(Here $\bar{\lambda}_k$ denotes the conjugate complex number of λ_k.)

Proof. The equality (8) has been proved in N. I. Achieser [1, Sect. 14] by Hilbert space methods. The inequality (9) has bene proved by the author [5, pp. 73–74] for real positive numbers q and λ_k. With little change this proof is also valid for complex numbers q and λ_k with positive real parts.

Let $1 \leqslant p < 2$, ϵ as above, and $\gamma = (2-p-2\epsilon)/(2p)$. Then, for any complex numbers a_k $(k = 1,..., s)$,

$$\tilde{E}_s(x^q; \Lambda)_p \leqslant \left\| x^q - \sum_{k=1}^{s} a_k x^{\lambda_k} \right\|_p = \left(\int_0^1 x^{-\gamma p} \left\| x^{q+\gamma} - \sum_{k=1}^{s} a_k x^{\lambda_k+\gamma} \right\|^p dx \right)^{1/p}$$
$$\leqslant \epsilon^{-(2-p)/(2p)} \left\| x^{q+\gamma} - \sum_{k=1}^{s} a_k x^{\lambda_k+\gamma} \right\|_2,$$

where we have applied Hölder's inequality. If we choose a_k $(k = 1,..., s)$ optimally and apply (8), we immediately obtain (11). The inequality (10) will follow from the next

LEMMA 3. *Let* $1 \leqslant r < p < +\infty$, $q > -1/p$, $q \neq 0$, $\operatorname{Re} \lambda_k > -1/p$, $\lambda_k \neq 0$ $(k = 1,..., s)$. *There exists a constant* $A(r,p)$ *depending only on r and p with the following property: for any complex coefficients* a_k $(k = 0, 1,..., s)$ *satisfying*

$$\sum_{k=0}^{s} a_k = 1, \tag{12}$$

the inequality

$$\left(\int_0^1 \left| x^q - a_0 - \sum_{k=1}^{s} a_k x^{\lambda_k} \right|^p dx \right)^{1/p}$$
$$\leqslant |q| A \left(\int_0^1 \left| x^{q+1/p-1/r} - \sum_{k=1}^{s} b_k x^{\lambda_k+1/p-1/r} \right|^r dx \right)^{1/r} \tag{13}$$

holds, where $b_k = a_k\lambda_k/q$ $(k = 1,..., s)$.

Proof. We denote

$$g(x) = x^q - a_0 - \sum_{k=1}^{s} a_k x^{\lambda_k}, \qquad h(x) = x^{q-1} - \sum_{k=1}^{s} b_k x^{\lambda_k - 1}.$$

Then, since $g(1) = 0$ and $g'(x) = qh(x)$,

$$I = \left(\int_0^1 | g(x)|^p \, dx\right)^{1/p} = | q | \left(\int_0^1 \left| \int_x^1 h(y) \, dy \right|^p dx\right)^{1/p}.$$

Let α denote a real number satisfying $1 - 1/r < \alpha < 1 - 1/r + 1/p$. (For example $\alpha = 1 - 1/r + 1/(2p)$.) Using Hölder's inequality for r and $r' = r/(r-1)$ we obtain

$$J(x) = \left| \int_x^1 h(y) \, dy \right| = \left| \int_x^1 y^{-\alpha}(y^\alpha h(y)) \, dy \right|$$
$$\leqslant K_1 x^{-\alpha + 1/r'} \left(\int_x^1 | y^\alpha h(y)|^r \, dy\right)^{1/r},$$

where

$$K_1 = \begin{cases} (\alpha r' - 1)^{-1/r'}, & \text{if } r > 1, \\ 1, & \text{if } r = 1. \end{cases}$$

Therefore,

$$I \leqslant | q | K_1 \left(\int_0^1 \left\{\int_x^1 x^{r-1-r\alpha} | y^\alpha h(y)|^r \, dy\right\}^{p/r} dx\right)^{1/p}. \tag{14}$$

In (14) we apply for $p^* = p/r$ and

$$\varphi(x, y) = \begin{cases} x^{r-1-r\alpha} | y^\alpha h(y)|^r, & \text{if } x \leqslant y \leqslant 1, \\ 0, & \text{if } 0 \leqslant y < x, \end{cases}$$

the generalized Minkowski inequality for integrals, i.e.,

$$\left(\int_0^1 \left| \int_0^1 \varphi(x, y) \, dy \right|^{p^*} dx\right)^{1/p^*} \leqslant \int_0^1 \left\{\int_0^1 | \varphi(x, y)|^{p^*} dx\right\}^{1/p^*} dy, \tag{15}$$

$p^* \geqslant 1$ (cf. N. I. Achieser [1, Sect. 5]). Then,

$$I \leqslant | q | K_1 \left(\int_0^1 \left\{\int_0^1 | \varphi(x, y)|^{p/r} dx\right\}^{r/p} dy\right)^{1/r}$$
$$= | q | K_1 \left(\int_0^1 | y^\alpha h(y)|^r \left\{\int_0^y x^{(r-1-r\alpha)p/r} dx\right\}^{r/p} dy\right)^{1/r}.$$

Therefore, the inequality (13) follows immediately for

$$A = K_1(1 + (1 - \alpha - 1/r)\,p)^{-1/p}.$$

This concludes the proof of Lemma 3.

Now we can easily prove the inequality (10): For $2 < p < \infty$ and $r = 2$ we choose the coefficients b_k $(k = 1,\dots, s)$ in (13) optimally. Then we define

$$a_k = qb_k/\lambda_k\ (k = 1,\dots, s), \qquad a_0 = 1 - \sum_{k=1}^{s} a_k\,.$$

It follows from (13) that

$$E_s(x^q; \Lambda)_p \leqslant |\,q\,|\, A_p \inf_{b_k} \Big\| x^{q+1/p-1/2} - \sum_{k=1}^{s} b_k x^{\lambda_k+1/p-1/2} \Big\|_2\,. \tag{16}$$

If we choose $\alpha = (4 + p)/(4 + 2p)$, then

$$A_p = (2\alpha - 1)^{-1/2}(1 + (\tfrac{1}{2} - \alpha)\,p)^{-1/p} = (1 + p/2)^{1/2+1/p}.$$

In (16) we apply the equality (8) and obtain (10). Thus, the proof of Lemma 2 is complete.

Combining the inequality (3) with the results of Lemma 1 and 2 we have proved the following

THEOREM 1. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of distinct complex numbers with positive real parts. Let s and n be any positive integers. Then, for $f \in L^p[0, 1]$,*

$$E_s(f; \Lambda)_p \leqslant w_p(f; 1/n)\{C_p + D_p{}^* \cdot R_p(\epsilon) \cdot I_{ns}\}, \tag{17}$$

where

$$I_{ns} = \sum_{q=2}^{n} n^{q+1/p}(e/q)^q \prod_{k=1}^{s} \frac{|\,q - \lambda_k\,|}{|\,q + \bar{\lambda}_k + 2/p - d_p(\epsilon)|}\,, \tag{18}$$

$$R_p(\epsilon) = \begin{cases} 1, & \text{if } 2 \leqslant p \leqslant \infty, \\ \epsilon^{-(2-p)/(2p)}, & \text{if } 1 \leqslant p < 2, \end{cases} \qquad d_p(\epsilon) = \begin{cases} 0, & \text{if } 2 \leqslant p \leqslant \infty, \\ 2\epsilon/p, & \text{if } 1 \leqslant p < 2. \end{cases} \tag{19}$$

C_p and $D_p{}^*$ are absolute constants, and ϵ is any positive, sufficiently small number.

Proof. We apply the inequality (3) together with Lemmas 1–2 and use Stirling's formula: $q! > (2\pi)^{1/2}\, q^{q+1/2}e^{-q}$. We notice that $a_{1n} = 0$, as the polynomial P_n of Lemma 1 is even.

2. Upper Bounds for the Degree of Best Approximation

It seems to be impossible to give a reasonable general formula for the degree of best approximation $E_s(f; \Lambda)_p$ which is valid for all sequences Λ simultaneously. Therefore we will examine the most important types of sequences Λ separately. The proofs of these theorems, however, are very similar: we always apply Theorem 1, where for a given integer s an appropriate integer n has to be chosen. It will be very convenient to evaluate the products of (18) by the following

LEMMA 4. *Let q and $\operatorname{Re} \lambda_k$ $(k = 1,..., s)$ be positive. Then for any $\delta \geqslant 0$,*

$$\prod_{k=1}^{s} \frac{| q - \lambda_k |}{| q + \bar{\lambda}_k + \delta |} \leqslant \exp\left(-(2q + \delta) \sum_{k=1}^{s} \frac{\operatorname{Re} \lambda_k}{q^2 + | \lambda_k |^2 + \delta \operatorname{Re} \lambda_k}\right). \tag{20}$$

Proof. Let $\alpha_k = \operatorname{Re} \lambda_k$. Then,

$$\frac{| q - \lambda_k |}{| q + \bar{\lambda}_k + \delta |} \leqslant \left(\frac{q^2 + | \lambda_k |^2 - 2q\alpha_k}{q^2 + | \lambda_k |^2 + 2(q + \delta)\, \alpha_k}\right)^{1/2}.$$

We apply the inequality $(1 - x)/(1 + x) \leqslant e^{-2x}$, $x \geqslant 0$, with

$$x = (2q + \delta)\alpha_k/(q^2 + | \lambda_k |^2 + \delta\alpha_k)$$

and obtain (20).

(A) Let the sequence Λ of complex numbers with positive real parts satisfy the condition

$$| \lambda_k | \geqslant Mk, \qquad | \lambda_k |^2 \geqslant Nk \operatorname{Re} \lambda_k \quad (k = 1, 2,...), \tag{21}$$

where $M > 0$, $N > 0$ are given real constants.

LEMMA 5. *If* (21) *holds, there exists a constant $B_1(M, N)$ such that for all positive integers q and s, and* $0 \leqslant \delta \leqslant 2$,

$$\prod_{k=1}^{s} \frac{| q - \lambda_k |}{| q + \bar{\lambda}_k + \delta |} \leqslant B_1^{\delta} e^{3q/N} (q/M)^{(2q+\delta)/N}\, \varphi(s)^{-2q-\delta}, \tag{22}$$

where

$$\varphi(s) = \exp\left(\sum_{k=1}^{s} \frac{\operatorname{Re} \lambda_k}{| \lambda_k |^2}\right). \tag{23}$$

Proof. Let $\alpha_k = \text{Re}\,\lambda_k$. Applying (21) we obtain

$$\sum_{k=1}^{s} \left(\frac{\alpha_k}{|\lambda_k|^2} - \frac{\alpha_k}{q^2 + |\lambda_k|^2 + \delta\alpha_k} \right)$$

$$\leqslant \sum_{k=1}^{s} \frac{\alpha_k(q^2 + \delta\alpha_k)}{|\lambda_k|^2 (q^2 + |\lambda_k|^2)} \leqslant \sum_{k=1}^{s} \frac{q^2/(Nk) + \delta}{q^2 + (Mk)^2}$$

$$\leqslant (3/2 + \log(q/M))/N + \delta/q^2 + \delta\pi/(2Mq). \tag{24}$$

The inequality (22) follows immediately from Lemma 4 with $B_1 \leqslant \exp(4 + 3/(2N) + 2\pi/M)$.

We are led to the following by Lemma 5.

THEOREM 2. *Under the condition* (21) *there exists a constant* $K_A(p, M, N)$ *such that for any* $f \in L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, *and any* $s \geqslant 1$

$$E_s(f, \Lambda)_p \leqslant K_A w_p(f; \varphi(s)^{-N}), \qquad \text{if } 0 < N < 2, \tag{25}$$

$$E_s(f; \Lambda)_p \leqslant K_A w_p(f; (\log \varphi(s))^{\alpha_p} \varphi(s)^{-2}), \qquad \text{if } N \geqslant 2, \tag{26}$$

where

$$\alpha_p = \begin{cases} 0 & \text{if } 2 \leqslant p \leqslant \infty, \\ (2-p)/(2+4p) & \text{if } 1 \leqslant p < 2, \end{cases}$$

and $\varphi(s)$ *is defined by* (23).

Proof. Let K_j ($j = 1,..., 4$) denote positive numbers depending only on p, M, N.

(a) Let $0 < N < 2$. We choose $\epsilon = 1 - N/2$ and the integer n such that

$$n - 1 < K^{*-N/2}\varphi(s)^N \leqslant n, \qquad \text{where} \qquad K^* = 2e^{1+3/N}M^{-2/N}.$$

Then, we obtain from Theorem 1 and Lemma 5 (with $\delta = 2/p - d_p(\epsilon) \geqslant 0$)

$$I_{ns} \leqslant B_1{}^{\delta} M^{-\delta/N} \sum_{q=2}^{n} n^{q+1/p} q^{-q+(2q+\delta)/N} (K^*/2)^q\, \varphi(s)^{-2q-\delta}$$

$$\leqslant K_1 \sum_{q=2}^{n} q^{\delta/N} 2^{-q} \varphi(s)^{N/p-\delta} \leqslant K_2\,,$$

since $N/p - \delta \leqslant (N - 2 + 2\,\epsilon)/p = 0$.

Applying (17) and the property

$$w_p(f; vt) \leqslant (v+1)\, w_p(f; t), \qquad v \geqslant 0, \qquad t \geqslant 0, \tag{27}$$

of the L^p modulus of continuity, we obtain (25).

(b) Let $N \geqslant 2$. We choose $\epsilon = \min\{1; (\log \varphi(s))^{-1}\}$ and the integer n such that

$$n - 1 < K^{*-1} \epsilon^{\alpha_p} \varphi(s)^2 \leqslant n.$$

Then, from Theorem 1 and Lemma 5 (with $\delta = 2/p - d_p(\epsilon) \geqslant 0$)

$$I_{ns} \leqslant K_3 \sum_{q=2}^{n} q^{\delta/N} 2^{-q} \epsilon^{\alpha_p(q+1/p)} \varphi(s)^{d_p(\epsilon)}.$$

Since

$$\varphi(s)^{d_p(\epsilon)} \leqslant e^{2/p} \text{ and } \epsilon^{\alpha_p(q+1/p)} \leqslant \epsilon^{\alpha_p(2+1/p)} = (R_p(\epsilon))^{-1},$$

we have

$$I_{ns} \leqslant K_4 (R_p(\epsilon))^{-1} \tag{28}$$

and from (17), (28), and (27) we obtain the inequality (26).

Remark. If Λ is a real sequence, the condition (21) is equivalent to $\lambda_k \geqslant Nk$ $(k = 1, 2,...)$. Then $\varphi(s) = \exp(\sum_{k=1}^{s} 1/\lambda_k)$, and our Theorem 2 contains the main results of the above mentioned papers [2–4, 10, 14]. Compare also [5, 6].

(B) Let the sequence Λ of complex numbers with positive real parts satisfy the condition

$$|\lambda_k| \geqslant Mk, \qquad |\lambda_k|^2 \leqslant Nk \operatorname{Re} \lambda_k \quad (k = 1, 2,...), \tag{29}$$

where $0 < M \leqslant N < +\infty$ are given real constants.

LEMMA 6. *If* (29) *holds, there exists a constant* $B_2(M, N)$ *such that for all positive integers* q *and* s, *and* $0 \leqslant \delta \leqslant 2$,

$$\prod_{k=1}^{s} \frac{|q - \lambda_k|}{|q + \bar{\lambda}_k + \delta|} \leqslant B_2 \{q/(Ms)\}^{(2q+\delta)/N}. \tag{30}$$

Proof. Applying (29) we obtain

$$\sum_{k=1}^{s} \frac{\alpha_k}{q^2 + |\lambda_k|^2 + \delta \alpha_k} \geqslant \frac{1}{N} \int_1^s \frac{x\, dx}{(q/M)^2 + \{x + \delta/(2N)\}^2}$$

$$\geqslant \frac{1}{2N} \log \frac{(q/M)^2 + \{s + \delta/(2N)\}^2}{(q/M)^2 + \{1 + \delta/(2N)\}^2} - \delta M \pi/(4qN^2)$$

and Lemma 4 leads us immediately to the inequality (30).

From Lemma 6 we have the following.

THEOREM 3. *Under the condition* (29) *there exists a constant* $K_B(p, M, N)$ *such that for any* $f \in L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, *and any* $s \geqslant 1$

$$E_s(f; \Lambda)_p \leqslant K_B w_p(f; 1/s), \qquad \text{if } 0 < N < 2, \tag{31}$$

and

$$E_s(f; \Lambda)_p \leqslant K_B w_p(f; \{\log(s+1)\}^{\alpha_p} s^{-2/N}), \qquad \text{if } N \geqslant 2, \tag{32}$$

where

$$\alpha_p = \begin{cases} 0 & \text{if } 2 \leqslant p \leqslant \infty, \\ (2-p)/(2+4p) & \text{if } 1 \leqslant p < 2. \end{cases}$$

Proof. Let K_j ($j = 1,..., 4$) denote positive numbers depending only on p, M, N.

(a) Let $0 < N < 2$. We choose $\epsilon = 1 - N/2$ and the integer n such that $n - 1 < K^{*-N/2}s \leqslant n$, where $K^* = 2eM^{-2/N}$. Then, from Theorem 1 and Lemma 6 (with $\delta = 2/p - d_p(\epsilon) \geqslant 0$),

$$I_{ns} \leqslant B_2 M^{-\delta/N} \sum_{q=2}^{n} n^{q+1/p} q^{-q+(2q+\delta)/N} (K^*/2)^q s^{-(2q+\delta)/N}$$

$$\leqslant K_1 \sum_{q=2}^{n} q^{\delta/N} 2^{-q} s^{1/p-\delta/N} \leqslant K_2,$$

since $1/p - \delta/N \leqslant (1 - 2/N + 2\epsilon/N)/p = 0$. Therefore, the inequality (31) follows from (17) and (27).

(b) Let $N \geqslant 2$. We choose $\epsilon = \min\{1; (\log(s+1))^{-1}\}$ and the integer n such that

$$n - 1 < K^{*-1} \epsilon^{\alpha_p} s^{2/N} \leqslant n.$$

Then, from Theorem 1 and Lemma 6 (with $\delta = 2/p - d_p(\epsilon) \geqslant 0$)

$$I_{ns} \leqslant K_3 \sum_{q=2}^{n} q^{\delta/N} 2^{-q} \epsilon^{\alpha_p(q+1/p)} s^{d_p(\epsilon)/N}. \tag{33}$$

Consequently, we obtain

$$I_{ns} \leqslant K_4 (R_p(\epsilon))^{-1}, \tag{34}$$

since

$$s^{d_p(\epsilon)/N} \leqslant e^{2/(Np)} \text{ and } \epsilon^{\alpha_p(q+1/p)} \leqslant \epsilon^{\alpha_p(2+1/p)} = (R_p(\epsilon))^{-1}.$$

Then, the inequalities (17), (34), and (27) lead us to (32), and the proof of Theorem 3 is complete.

COROLLARY. *Let Λ be a real sequence satisfying*

$$Mk \leqslant \lambda_k \leqslant Nk \qquad (k = 1, 2, \ldots), \tag{35}$$

where $0 < M \leqslant N < +\infty$ are given real constants. Then inequality (31) *holds if $N < 2$ and inequality* (32) *holds if $N \geqslant 2$.*

Proof. For real numbers λ_k the condition (29) is equivalent to (35) and Theorem 3 is applicable.

(C) The sequences Λ in the preceding Theorems 2, 3 satisfy $|\lambda_k| \geqslant Mk$ ($k = 1, 2, \ldots$). Our method described by Theorem 1, however, is valid for any sequence Λ of complex numbers with positive real parts. As an example, for which the above property $|\lambda_k| \geqslant Mk$ does not hold, we now discuss complex sequences Λ with a finite limit point, i.e.,

$$\lim_{k\to\infty} \lambda_k = \lambda^*, \qquad \operatorname{Re}\lambda^* > 0. \tag{36}$$

LEMMA 7. *If* (36) *holds, there exist positive numbers B_3 and c depending only on Λ such that for all positive integers q and s, and $0 \leqslant \delta \leqslant 2$,*

$$\prod_{k=1}^{s} \frac{|q - \lambda_k|}{|q + \bar{\lambda}_k + \delta|} \leqslant B_3 e^{-cs/q}. \tag{37}$$

Proof. Let $\alpha^* = \operatorname{Re}\lambda^*$. There exists an integer k_0 such that $\alpha_k = \operatorname{Re}\lambda_k \geqslant \alpha^*/2$ and $|\lambda_k| \leqslant 2|\lambda^*|$ for all $k > k_0$. Applying Lemma 4, we obtain for all $s \geqslant 2k_0$

$$\prod_{k=1}^{s} \frac{|q - \lambda_k|}{|q + \bar{\lambda}_k + \delta|} \leqslant \prod_{k_0+1}^{s} \frac{|q - \lambda_k|}{|q + \bar{\lambda}_k|} \leqslant \exp\left(-2q \sum_{k_0+1}^{s} \frac{\alpha_k}{q^2 + |\lambda_k|^2}\right)$$

$$\leqslant \exp(-q(s - k_0)\,\alpha^*/(q^2 + 4|\lambda^*|^2)) \leqslant e^{-cs/q},$$

where $c \leqslant \alpha^*/(2 + 8|\lambda^*|^2)$. Therefore, (37) holds for all $s \geqslant 1$.

THEOREM 4. *Under the condition* (36) *there exists a constant K_C depending only on Λ and p such that for any $f \in L^p[0, 1]$, $1 \leqslant p \leqslant \infty$, and any $s \geqslant 1$*

$$E_s(f; \Lambda)_p \leqslant K_C w_p(f; s^{-1/2}). \tag{38}$$

Proof. We choose $\epsilon = 1$ and the integer n such that

$$n - 1 < \{cs/2\}^{1/2} \leqslant n.$$

Then, from Theorem 1 and Lemma 7 (with $\delta = 2/p - d_p(\epsilon) \geqslant 0$),

$$I_{ns} \leqslant B_3 \sum_{q=2}^{n} n^{q+1/p}(e/q)^q e^{-cs/q} \leqslant B_3',$$

where B_3' depends only on Λ. Therefore the inequalities (17) and (27) lead us directly to (38), which concludes the proof of Theorem 4.

3. Lower Bounds for the Degree of Best Approximation

We now want to show that the upper bounds obtained in Theorems 2, 3 are essentially best possible. (We conjecture that the upper bounds of Theorem 4 for converging sequences Λ are also best possible, though we cannot prove it.) No inverse theorems are given. Instead, we either test our results by special functions f or apply some results of the theory of widths.

Lemma 8. *Let Λ be a sequence of complex numbers with real parts exceeding $-1/p$. Then for any real number $q > -1/p$, $q \neq 0$, there exists a number $C(p, q)$ depending only on p and q such that for any $s \geqslant 1$*

$$E_s(x^q; \Lambda)_p \geqslant C \prod_{k=1}^{s} \frac{| q - \lambda_k |}{| q + \bar{\lambda}_k + 2/p |} \qquad 1 \leqslant p \leqslant 2 \tag{39}$$

and

$$E_s(x^q; \Lambda)_p \geqslant C\epsilon^{(p-2)/(2p)} \prod_{k=1}^{s} \frac{| q - \lambda_k |}{| q + \bar{\lambda}_k + 2/p + \epsilon |} \qquad 2 < p \leqslant \infty, \tag{40}$$

where ϵ is any real number with $0 < \epsilon \leqslant 1$.

Proof. (a) Let $1 \leqslant p < 2$. For $\lambda_{s+1} = 0$ we obtain from Lemma 3 (after simple substitutions)

$$\left\| x^{q-1/2+1/p} - a_0 - \sum_{k=1}^{s+1} a_k x^{\lambda_k - 1/2+1/p} \right\|_2$$

$$\leqslant | q - 1/2 + 1/p \mid A(p, 2) \left\| x^q - \sum_{k=1}^{s+1} b_k x^{\lambda_k} \right\|_p .$$

We are led to the inequality (39), if we choose b_k ($k = 1,..., s + 1$) optimally and apply (8).

(b) Let $2 < p \leqslant \infty$. For any complex numbers a_k, $\alpha = 1 - \epsilon - 2/p$, and $\lambda_0 = 0$ we have

$$\left\| x^{q-\alpha/2} - \sum_{k=0}^{s} a_k x^{\lambda_k - \alpha/2} \right\|_2 = \left(\int_0^1 x^{-\alpha} \left| x^q - \sum_{k=0}^{s} a_k x^{\lambda_k} \right|^2 dx \right)^{1/2}$$
$$\leqslant (1 - \alpha r')^{-1/(2r')} \left\| x^q - \sum_{k=0}^{s} a_k x^{\lambda_k} \right\|_p,$$

where we have applied Hölder's inequality for $r = p/2$ and $r' = p/(p-2)$. Since $1 - \alpha r' = \epsilon p/(p-2)$, we obtain the inequality (40) if we choose a_k $(k = 0,..., s)$ optimally and apply (8).

In our next theorem we will apply Lemma 8 and demonstrate that the upper bounds obtained in Theorem 2 for $N \geqslant 2$ are best or almost best possible, at least for the functions $g(x) = x^q$, $0 < q + 1/p < 1$.

THEOREM 5. *Let Λ satisfy* (21) *for an $N \geqslant 2$. Let q be a real number with $0 < q + 1/p < 1$. Then for the function $g(x) = x^q$, $q \neq 0$, $q \notin \Lambda$,*

$$E_s(g; \Lambda)_p \geqslant C_0 \{\log \varphi(s)\}^{-\beta_p} w_p(g; \varphi(s)^{-2}), \tag{41}$$

where

$$\beta_p = \begin{cases} 0, & \text{if } 1 \leqslant p \leqslant 2, \\ (p-2)/(2p), & \text{if } 2 < p \leqslant \infty, \end{cases}$$

and C_0 depends only on p, q, and Λ.

Proof. (a) As $|\lambda_k| \geqslant Mk$, there exists an integer k_0 (depending on M) such that for all $k \geqslant k_0$, $|\lambda_k| \geqslant 10$ and, consequently,

$$|\lambda_k|^2 - (4q + 2\delta)\,\alpha_k - 8 > 0,$$

where

$$\delta = \begin{cases} 2/p, & \text{if } 1 \leqslant p \leqslant 2, \\ 2/p + \epsilon, & \text{if } 2 < p \leqslant \infty, \end{cases}$$

$\epsilon > 0$ sufficiently small. Then we have

$$\prod_{k=1}^{s} \frac{|q - \lambda_k|}{|q + \bar{\lambda}_k + \delta|} \geqslant C_1 \left(\prod_{k=k_0}^{s} \frac{|\lambda_k|^2 - (4q + 2\delta)\,\alpha_k - 8}{|\lambda_k|^2} \right)^{1/2}$$
$$\geqslant C_2 \exp\left(\frac{1}{2} \sum_{k=k_0}^{s} \log(1 - (4q + 2\delta)\,\alpha_k / |\lambda_k|^2) \right)$$
$$\geqslant C_3 \varphi(s)^{-2q-\delta}, \tag{42}$$

if we apply (in the last inequality) the property $|\lambda_k|^2 \geqslant Nk\alpha_k$, where $N \geqslant 2$ and C_1, C_2, C_3 are positive numbers depending only on p, q, and Λ.

(b) For $0 < q + 1/p < 1$, $q \neq 0$, $1 \leqslant p \leqslant \infty$, we notice that the L^p modulus of continuity of $g(x) = x^q$ satisfies

$$w_p(g; t) \leqslant C_4 t^{q+1/p}, \qquad 0 \leqslant t \leqslant 1, \tag{43}$$

for a positive number C_4, which depends only on p and q. Therefore, if $1 \leqslant p \leqslant 2$, we obtain from (39), (42), and (43) for $\delta = 2/p$ the inequality (41). If $2 < p \leqslant \infty$, we choose $\epsilon = \{\log \varphi(s)\}^{-1}$, $\delta = \epsilon + 2/p$. Then we obtain the inequality (41) from (40), (42), and (43), which completes the proof of Theorem 5.

We have demonstrated in Theorem 5 that for each sequence Λ satisfying (21) with an $N \geqslant 2$ we can find functions $g(x) = x^q$, for which the upper bounds (26) of Theorem 2 are best or almost best possible. However, it is easy to find sequences Λ satisfying the condition (21) with $0 < N < 2$ or (29) with $N \geqslant 2$, for which the upper bounds (25) of Theorem 2 or (32) of Theorem 3 are not best possible. The reason is that these conditions (i.e., (21) with $0 < N < 2$ and (29) with $N \geqslant 2$) are still too general. Therefore we are content to show that the upper bounds (25) and (32) are best possible at least for the special sequences Λ^* as follows.

Let Λ^* satisfy

$$|\lambda_k| \geqslant Mk, \qquad |\lambda_k|^2 = Nk \operatorname{Re} \lambda_k \quad (k = 1, 2,...). \tag{44}$$

Then the conditions (21) and (29) are satisfied. We have

$$\varphi(s) = \exp\left(\sum_{k=1}^{s} \frac{\operatorname{Re} \lambda_k}{|\lambda_k|^2}\right) \approx s^{1/N}. \tag{45}$$

Therefore, if $N \geqslant 2$, the upper bounds of (26) and (32) are identical and (32) cannot be improved in the sense of Theorem 5. If $0 < N < 2$, the inequalities (25) and (31) are identical, i.e.,

$$E_s(f; \Lambda^*)_p \leqslant K_{A,B} w_p(f; 1/s). \tag{46}$$

Finally, from results of the theory of widths we realize that the "rate of convergence $1/s$" in (46) for Λ^* and in (31) for general sequences Λ is best possible in the function classes Lip $_c(\alpha, p)$ (i.e., the complex analog of Lip(α, p)). We only have to consider the real and imaginary parts of the functions f and the Λ-polynomials and apply the following.

LEMMA 9. *Let* $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant \infty$. *We denote* $A = \text{Lip}(\alpha, p) = \{f \in L^p[0, 1] \mid f$ *real valued*, $w_p(f; t) \leqslant t^\alpha \ (0 \leqslant t \leqslant 1)\}$. *Then the sth widths of the classes A are*

$$d_s(A) \approx s^{-\alpha}, \tag{47}$$

where the sth width is defined by

$$d_s(A) = \inf_{X_s} \sup_{f \in A} \{ \inf_{P \in X_s} \|f - P\|_p \}, \tag{48}$$

and X_s denotes any subspace of the real $L^p[0, 1]$ space of dimension s.

Proof. The proof of (47) for $p = \infty$ and further definitions and properties of the width are described in G. G. Lorentz [11, Chap. 9]. If $1 \leqslant p < \infty$, we combine [12, Theorems 10 and 6 (inequality (4))] of G. G. Lorentz and obtain

$$d_s(A) \geqslant Ks^{-\alpha} \qquad (K \text{ is a positive constant}).$$

The estimate of $d_s(A)$ from above follows, for instance, from (4) or (31).

Notes. 1. The method described in Theorem 1 also provides upper bounds for the degree of best approximation for differentiable functions. For more information see the author's paper [6], where this problem has been discussed in great detail for real sequences Λ.

2. Recently, the author [7] has announced results on Jackson–Müntz theorems for intervals $[a, 1]$, $a > 0$. The details including complex exponents Λ have been published in [19]. For positive intervals, the "singular" point $x = 0$ has less influence. Therefore the approximation properties of many sequences Λ are much better than for the interval [0, 1]. Substituting

$$x = e^{t-B}, \qquad t \in [A, B], \qquad x \in [a, 1],$$

we are led to the interesting equivalent problem where functions $F \in C[A, B]$ or $F \in L^p[A, B]$, $[A, B]$ finite, are to be approximated by linear exponential sums $\sum_{k=1}^{s} a_k e^{\lambda_k t}$.

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